Chapter 2

• Functions and Graphs
2.1

• Basics of Functions & Their Graphs
Objectives

- Find the domain & range of a relation.
- Evaluate a function.
- Graph functions by plotting points.
- Obtain information from a graph.
- Identify the domain & range from a graph.
- Identify x-y intercepts from a graph.
Domain & Range

• Domain: first components in the relation (independent variable or \(x\)-values)
• Range: second components in the relation (dependent variable, the value depends on what the domain value is, aka \(y\)-values)
• Functions are SPECIAL relations: A domain element corresponds to exactly ONE range element.
EXAMPLE

• Consider the function: eye color
• (Assume all people have only one color)
• It IS a function because when asked the eye color of each person, there is only one answer.
• e.g. {(Joe, brown), (Mo, blue), (Mary, green), (Ava, brown), (Natalie, blue)}
• NOTE: the range values are not necessarily unique.
Evaluating a function

• Common notation: \( f(x) = \text{function} \)
• Evaluate the function at various values of \( x \), represented as: \( f(a), f(b), \text{etc.} \)
• Example: \( f(x) = 3x - 7 \)
  \[
  f(2) = 3(2) - 7 = 6 - 7 = -1 \\
  f(3 - x) = 3(3 - x) - 7 = 9 - 3x - 7 = 2 - 3x
  \]
Graphing a functions

- Horizontal axis: x values
- Vertical axis: y values
- Plot points individually or use a graphing utility (calculator or computer algebra system)
- Example: \[ y = x^2 + 1 \]
Table of function values

\[ y = x^2 + 1 \]

<table>
<thead>
<tr>
<th>X (domain)</th>
<th>Y (range)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4</td>
<td>17</td>
</tr>
<tr>
<td>-3</td>
<td>10</td>
</tr>
<tr>
<td>-2</td>
<td>5</td>
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<td>-1</td>
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<td>0</td>
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<td>3</td>
<td>10</td>
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<tr>
<td>4</td>
<td>17</td>
</tr>
</tbody>
</table>
Graphs of functions
Can you identify domain & range from the graph?

- Look horizontally. What all x-values are contained in the graph? That’s your domain!
- Look vertically. What all y-values are contained in the graph? That’s your range!
What is the domain & range of the function with this graph?

1) Domain \((-\infty, \infty)\), Range \((-\infty, \infty)\)
2) Domain \((-3, \infty)\), Range \((-\infty, \infty)\)
3) Domain \((-3, \infty)\), Range \((-3, \infty)\)
4) Domain \((-\infty, \infty)\), Range \((-3, \infty)\)

Correct Answer: 4
Finding intercepts:

- **X-intercept**: where the function crosses the x-axis. What is true of every point on the x-axis? The y-value is ALWAYS zero.
- **Y-intercept**: where the function crosses the y-axis. What is true of every point on the y-axis? The x-value is ALWAYS zero.
- Can the x-intercept and the y-intercept ever be the same point? YES, if the function crosses through the origin!
2.2

• More of Functions and Their Graphs
Objectives

• Find & simplify a function’s difference quotient.
• Understand & use piecewise functions.
• Identify intervals on which a function increases, decreases, or is constant.
• Use graphs to locate relative maxima or minima.
• Identify even or odd functions & recognize the symmetries.
Difference Quotient

- Useful in discussing the rate of change of function over a period of time
- EXTREMELY important in calculus, (h represents the difference in two x values)

\[
\frac{f(x + h) - f(x)}{h}
\]
Find the difference quotient

\[ f(x) = 2x^3 - 2x + 1 \]
\[ f(x + h) = 2(x + h)^3 - 2(x + h) + 1 \]
\[ f(x + h) = 2(x^3 + 3x^2h + 3xh^2 + h^3) - 2x - 2h + 1 \]
\[ f(x + h) = 2x^3 + 6x^2h + 6xh^2 + 2h^3 - 2x - 2h + 1 \]

\[ \frac{f(x + h) - f(x)}{h} = \frac{2x^3 + 6x^2h + 6xh^2 + 2h^3 - 2x - 2h + 1 - (2x^3 - 2x + 1)}{h} \]
\[ \frac{f(x + h) - f(x)}{h} = \frac{6x^2h + 6xh^2 + 2h^3 - 2h}{h} = \frac{h(6x^2 + 6xh + 2h^2 - 2)}{h} \]
\[ \frac{f(x + h) - f(x)}{h} = 6x^2 + 6xh + 2h^2 - 2 \]
What is a piecewise function?

• A function that is defined differently for different parts of the domain.

• Examples: You are paid $10/hr for work up to 40 hrs/wk and then time and a half for overtime.

\[
f(x) = \begin{cases} 
10x & \text{if } x \leq 40 \\
15x & \text{if } x > 40 
\end{cases}
\]
Increasing and Decreasing Functions

- **Increasing**: Graph goes “up” as you move from left to right.
  \[ x_1 < x_2, \quad f(x_1) < f(x_2) \]

- **Decreasing**: Graph goes “down” as you move from left to right.
  \[ x_1 < x_2, \quad f(x_1) > f(x_2) \]

- **Constant**: Graph remains horizontal as you move from left to right.
  \[ x_1 < x_2, \quad f(x_1) = f(x_2) \]
Even & Odd Functions

• Even functions are those that are mirrored through the y-axis. (if –x replaces x, the y value remains the same) (e.g. 1st quadrant reflects into the 2nd quadrant)

• Odd functions are those that are rotated through the origin. (if –x replaces x, the y value becomes –y) (e.g. 1st quadrant reflects into the 3rd quadrant)
Determine if the function is even, odd, or neither.

\[ f(x) = 2(x - 4)^2 - 2x^2 \]

1. Even
2. Odd
3. Neither

Correct Answer: 3
2.3

• Linear Functions & Slope
Objectives

• Calculate a line’s slope.
• Write point-slope form of a line’s equation.
• Model data with linear functions and predict.
What is slope? The steepness of the graph, the rate at which the y values are changing in relation to the changes in x.

How do we calculate it?

\[ \text{slope} = m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \]
A line has one slope

- Between any 2 pts. on the line, the slope MUST be the same.
- Use this to develop the point-slope form of the equation of the line.

\[ y - y_1 = m(x - x_1) \]

- Now, you can develop the equation of any line if you know either a) 2 points on the line or b) one point and the slope.
Find the equation of the line that goes through (2,5) and (-3,4)

1\textsuperscript{st}: Find slope of the line

\[ m = \frac{5 - 4}{2 - (-3)} = \frac{1}{5} \]

2\textsuperscript{nd}: Use either point to find the equation of the line & solve for \( y \).

\[ y - 5 = \frac{1}{5} (x - 2) \]

\[ y = \frac{1}{5} x - \frac{2}{5} + 5 = \frac{1}{5} x + \frac{3}{5} \]
2.5 Transformation of Functions

- Recognize graphs of common functions
- Use vertical shifts to graph functions
- Use horizontal shifts to graph functions
- Use reflections to graph functions
- Graph functions w/ sequence of transformations
• **Vertical shifts**
  – Moves the graph up or down
  – Impacts only the “y” values of the function
  – No changes are made to the “x” values

• **Horizontal shifts**
  – Moves the graph left or right
  – Impacts only the “x” values of the function
  – No changes are made to the “y” values
Recognizing the shift from the equation. Examples of shifting the function $f(x) = x^2$

- Vertical shift of 3 units up

$$f(x) = x^2, h(x) = x^2 + 3$$

- Horizontal shift of 3 units left (HINT: x’s go the opposite direction that you might believe.)

$$f(x) = x^2, g(x) = (x + 3)^2$$
Combining a vertical & horizontal shift

- Example of function that is shifted down 4 units and right 6 units from the original function.

\[ f(x) = |x|, \quad g(x) = |x - 6| - 4 \]
Reflecting

• Across x-axis (y becomes negative, \(-f(x)\))

• Across y-axis (x becomes negative, \(f(-x)\))
2.6 Combinations of Functions; Composite Functions

• Objectives
  – Find the domain of a function
  – Form composite functions.
  – Determine domains for composite functions.
  – Write functions as compositions.
Using basic algebraic functions, what limitations are there when working with real numbers?

A) You can never divide by zero. Any values that would result in a zero denominator are NEVER allowed, therefore the domain of the function (possible x values) would be limited.

B) You cannot take the square root (or any even root) of a negative number. Any values that would result in negatives under an even radical (such as square roots) result in a domain restriction.
Example

• Find the domain \( \frac{\sqrt{x-2}}{x^2 - 5x + 6} \)

• There are x’s under an even radical AND x’s in the denominator, so we must consider both of these as possible limitations to our domain.

\[
x - 2 \geq 0, \quad x \geq 2 \\
x^2 - 5x + 6 \neq 0 \\
(x - 3)(x - 2) \neq 0, \quad x \neq 2, 3
\]

**Domain** : \( \{ x : x > 2, \ x \neq 3 \} \)
Composition of functions

• Composition of functions means the output from the inner function becomes the input of the outer function.

• f(g(3)) means you evaluate function g at x=3, then plug that value into function f in place of the x.

• Notation for composition:

\[(f \circ g)(x) = f(g(x))\]
2.7 Inverse Functions

• Objectives
  – Verify inverse functions
  – Find the inverse of a function.
  – Use the horizontal line test to determine one-to-one.
  – Given a graph, graph the inverse.
  – Find the inverse of a function & graph both functions simultaneously.
What is an inverse function?

- A function that “undoes” the original function.
- A function “wraps an x” and the inverse would “unwrap the x” resulting in x when the 2 functions are composed on each other.

\[ f(f^{-1}(x)) = f^{-1}(f(x)) = x \]
How do their graphs compare?

- The graph of a function and its inverse always mirror each other through the line $y=x$.
- Example: $y = \frac{1}{3}x + 2$ and its inverse $= 3(x-2)$
- Every point on the graph $(x,y)$ exists on the inverse as $(y,x)$ (i.e. if $(-6,0)$ is on the graph, $(0,-6)$ is on its inverse.)
Do all functions have inverses?

• Yes, and no. Yes, they all will have inverses, BUT we are only interested in the inverses if they ARE A FUNCTION.

• DO ALL FUNCTIONS HAVE INVERSES THAT ARE FUNCTIONS? NO.

• Recall, functions must pass the vertical line test when graphed. If the inverse is to pass the vertical line test, the original function must pass the HORIZONTAL line test (be one-to-one)!
How do you find an inverse?

• “Undo” the function.
• Replace the x with y and solve for y.