CHAPTER 6

MATRICES AND DETERMINANTS
6.1 Matrix Solutions to Linear Systems

• Objectives
  – Write the augmented matrix for a linear system
  – Perform matrix row operations
  – Use matrices & Gaussian elimination to solve systems
  – Use matrices & Gauss-Jordan elimination to solve systems
What is a matrix?

• A set of numbers in rows & columns
• m x n describes the dimensions of the matrix (m rows & n columns)
• The matrix is contained within brackets.
• Example of a 3 x 3 matrix:

\[
\begin{bmatrix}
4 & 2 & 5 \\
1 & -4 & 3 \\
0 & -3 & 8
\end{bmatrix}
\]
Solving a system of 3 equations with 3 variables

• Graphically, you are attempting to find where 3 planes intersect.
• If you find 3 numeric values for \((x,y,z)\), this indicates the 3 planes intersect at that point.
Representing a system of equations in a matrix

• If a linear system of 3 equations involved 3 variables, each column represents the different variables & constant, and each row represents a separate equation.

• Example: Write the following system as a matrix

\[
\begin{align*}
2x + 3y - 3z &= 7 \\
5x + y - 4z &= 2 \\
4x + 2y - z &= 6
\end{align*}
\]

\[
\begin{bmatrix}
2 & 3 & -3 & 7 \\
5 & 1 & -4 & 2 \\
4 & 2 & -1 & 6
\end{bmatrix}
\]
Solving linear systems using Gaussian elimination

• Use techniques learned previously to solve equations (addition & substitution) to solve the system
• Variables are eliminated, but each column represents a different variable
• Perform addition &/or multiplication to simplify rows. Have one row contain 2 zeros, a one, and a constant. This allows you to solve for one variable.
• Work up the matrix and solve for the remaining variables.
A matrix should look like this after Gaussian elimination is applied

\[
\begin{bmatrix}
1 & y_1 & z_1 & k_1 \\
0 & 1 & z_2 & k_2 \\
0 & 0 & 1 & k_3 \\
\end{bmatrix}
\]
Examples 3 & 4 (p. 541-543)

• These examples outline a stepwise approach to use Gaussian elimination.
• One row (often the bottom row) will contain \((0\ 0\ 0\ 1\ \text{constant})\) using this method.
• After the last variable is solved in the bottom equation, substitute in for that variable in the remaining equations.
Question: Must the last row contain (0 0 0 1 constant)?

- Yes, in Gaussian elimination, but there are other options.
- Look again at example 3, page 539.
- If you multiply the first row by (-1) and add it to the 2\textsuperscript{nd} row, the result is (-2 0 0 -12)
- What does this mean? -2x = -12, x=6
- By making an informed decision as to what variables to eliminate, we solved for a variable much more quickly!
- Next, multiply row 1 by (-1) & add to 3\textsuperscript{rd} row: (-2 2 0 6). Recall x=-6, therefore this becomes
  - -2(6)+2y=6, y=3
  - Now knowing x & y, solve for z in any row (row 2?)
  - 6 + 3 + 2z = 19, z=5    Solution: (6,3,5)
If this method is quicker, why would we use Gaussian elimination?

• Using a matrix to solve a system by Gaussian elimination provides a standard, **programmable** approach.

• When computer programs (may be contained in calculators) solve systems. This is the method utilized!
6.2 Inconsistent & Dependent Systems & Their Applications

• Objectives
  – Apply Gaussian elimination to systems without unique solutions
  – Apply Gaussian elimination to systems with more variables than equations
  – Solve problems involving systems without unique solutions.
How would you know, with Gaussian elimination, that there are no solutions to your system?

• When reducing your matrix (attempting to have rows contain only 0’s, 1’s & the constant) a row becomes 0 0 0 0 k
• What does that mean? Can you have 0 times anything equal to a non-zero constant? NO! No solution!
• Inconsistent system – no solution
Graphically, what is happening with an inconsistent system?

• Recall, with 3 variables, the equation represents a plane, therefore we are considering the intersection of 3 planes.

• If a system is inconsistent, 2 or more of the planes may be parallel OR 2 planes could intersect forming 1 line and a different pair of planes intersect at a different line, therefore there is nothing in common to all three planes.
Could there be more than one solution?

• Yes! If the planes intersect to form a line, rather than a point, there would be infinitely many solutions. All pts. lying on the line would be solutions.

• You can’t state infinitely many points, so you state the general form of all points on the line, in terms of one of the variables.
What if your system has 3 variables but only 2 equations?

• Graphically, this is the intersection of 2 planes.
• 2 planes cannot intersect in 1 point, rather they intersect in 1 line. (or are parallel, thus no solution)
• The solution is all points on that line.
• The ordered triple is represented as one of the variables (usually z) and the other 2 as functions of that variable: ex: (z+2,3z,z)
Dependent system

• Notice when there were infinitely many solutions, two variables were stated in terms of the 3rd. In other words, the x & y values are dependent on the value selected for z.

• If there are infinitely many solutions, the system is considered to be dependent.
6.5 Determinants & Cramer’s Rule

• Objectives
  – Evaluate a 2\textsuperscript{nd}-order determinant
  – Solve a system of linear equations in 2 variables using Cramer’s rule
  – Evaluate a 3\textsuperscript{rd}-order determinant
  – Solve a system of linear equations in 3 variables using Cramer’s rule
  – Use determinants to identify inconsistent & dependent systems
  – Evaluate higher-order determinants
Determinant of a 2x2 matrix

• If $A$ is a matrix, the determinant is $\det(A)$

$$A = \begin{bmatrix} 3 & -2 \\ 4 & 5 \end{bmatrix}, \det(A) = \begin{vmatrix} 3 & -2 \\ 4 & 5 \end{vmatrix} = 3(5) - (-2)4 = 23$$
When are determinants useful?

- They can be used to solve a system of equations
- Cramer’s Rule

\[
\begin{align*}
    a_1 x + b_1 y &= c_1 \\
    a_2 x + b_2 y &= c_2
\end{align*}
\]

\[
x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}
\]
Finding a determinant of a 3x3 matrix

- More complicated, but it can be done!

\[
\begin{vmatrix}
a_1 & b_1 & c_1 \\
a_2 & b_2 & c_2 \\
a_3 & b_3 & c_3 \\
\end{vmatrix}
= a_1 \cdot \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \cdot \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \cdot \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}
\]

- It’s often easier to pick your “home row/column” (the one with the multipliers) to be a row/column that has one or more zeros in it.
Determinants can be used to solve a linear system in 3 variables

**CRAMER'S RULE**

\[ x = \frac{D_x}{D}, \quad y = \frac{D_y}{D}, \quad z = \frac{D_z}{D} \]
What is D and $D_x, D_y, D_z$

- D is the determinant that results from the coefficients of all variables.

- $D_x$ is the determinant that results when each x coefficient is replaced with the given constants.

- $D_y$ is the determinant that results when each y coefficient is replaced with the given constants.

- $D_z$ is the determinant that results when the z coefficients are replaced with the given constants.
Find z, given

\[2x + y = 7\]
\[-x + 3y + z = 5\]
\[3x + 2y - 4z = 10\]