CHAPTER 8

Sequences, Induction, & Probability
8.1 Sequences & Summation Notation

• Objectives
  – Find particular terms of sequence from the general term
  – Use recursion formulas
  – Use factorial notation
  – Use summation notation
What is a sequence?

• An infinite sequence is a function whose domain is the set of positive integers. The function values, terms, of the sequences are represented by \( a_1, a_2, a_3, \ldots a_n \ldots \)

• Sequences whose domains are the first \( n \) integers, not ALL positive integers, are finite sequences.
Recursive Sequences

- A specific term is given.
- Other terms are determined based on the value of the previous term(s)
- Example: \( a_3 = 10, a_{n+1} = 3 \cdot a_n + 1 \), find \( a_1, a_2, a_4 \)

\[
\begin{align*}
a_3 &= 10 = 3a_2 + 1 \\
a_2 &= \frac{10 - 1}{3} = 3 = 3a_1 + 1 \\
a_1 &= \frac{3 - 1}{3} = \frac{2}{3} \\
a_4 &= 3a_3 + 1 = 3(10) + 1 = 31
\end{align*}
\]
Find the 1st 3 terms of the sequence:

\[ a_n = \frac{n + 1}{n!} \]

• 1) 4, 5/2, 6
• 2) 4, 5/2, 1
• 3) 1, 2, 3
• 4) 4, 5, 6
Summation Notation

- The sum of the first $n$ terms, as $i$ goes from 1 to $n$ is given as: 
  \[ \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \ldots + a_{n-1} + a_n \]

- Example:

\[
\sum_{i=4}^{8} 5i - 2 = [(5 \cdot 4) - 2] + [(5 \cdot 5) - 2] + [(5 \cdot 6) - 2] + [(5 \cdot 7) - 2] + [(5 \cdot 8) - 2] \\
= 18 + 23 + 28 + 33 = 102
\]
8.2 Arithmetic Sequences

• Objectives
  – Find the common difference for an arithmetic sequence
  – Write terms of an arithmetic sequence
  – Use the formula for the general term of an arithmetic sequence
  – Use the formula for the sum of the first $n$ terms of an arithmetic sequence
What is an arithmetic sequence?

• A sequence where there is a common difference between every 2 terms.

• Example: 5,8,11,14,17,…..

• The common difference (d) is 3

• If a specific term is known and the difference is known, you can determine the value of any term in the sequence

• For the previous example, find the 20\text{th} term
Example continued

• The first term is 5 and d=3
• Notice between the 1st & 2nd terms there is 1 (3). Between the 1st & 4th terms there are 3 (3’s). Between the 1st & nth terms there would be (n-1) 3’s
• 20th term would be the 1st term + 19(3’s)

\[ a_{20} = 5 + 19(3) = 62 \]
The sum of the 1\textsuperscript{st} n terms of an arithmetic sequence

- Since every term is increasing by a constant (d), the sequence, if plotted on a graph (x=the indicated term, y=the value of that term), would be a line with slope= d
- The average of the 1\textsuperscript{st} & last terms would be greater than the 1\textsuperscript{st} term by k and less than the last term by k. The same is true for the 2\textsuperscript{nd} term & the 2\textsuperscript{nd} to last term, etc
- Therefore, you can find the sum by replacing each term by the average of the 1\textsuperscript{st} & last terms (continue next slide)
Sum of an arithmetic sequence

• If there are \( n \) terms in the arithmetic sequence and you replace all of them with the average of the 1\(^{st} \) & last, the result is:

\[
S_n = n \cdot \left( \frac{a_1 + a_n}{2} \right)
\]
Find the sum of the 1st 30 terms of the arithmetic sequence if

\[ a_1 = -6, \quad d = 6 \]

- 1) 81
- 2) 3430
- 3) 2430
- 4) 168

» Answer: sum = 2430
8.3 Geometric Sequences & Series

• Objectives
  – Find the common ratio of a geometric sequence
  – Write terms of a geometric sequence
  – Use the formula for the general term of a geometric sequence
  – Use the formula for the sum of the 1st $n$ terms of a geometric sequence
  – Find the value of an annuity
  – Use the formula for the sum of an infinite geometric series
What is a geometric sequence?

• A sequence of terms that have a common multiplier ($r$) between all terms
• The multiplier is the ratio between the $(n+1)$th term & the $n$th term
• Example: -2,4,-8,16,-32,…
• The ratio between any 2 terms is ($-2$) which is the value you multiply any term by to find the next term
Given a term in a geometric sequence, find a specified other term

• Example: If 1\textsuperscript{st} term=3 and r=4, find the 14\textsuperscript{th} term
• Notice to find the 2\textsuperscript{nd} term, you multiply 3(4)
• To find the 3\textsuperscript{rd} term, you would multiply 3(4)(4)
• To find the 4\textsuperscript{th} term, multiply 3(4)(4)(4)
• To find the nth term, multiply: 3(4)(4)(4)… (n-1 times)
• 14\textsuperscript{th} term = 3(4)^{13} = 201,326,592

• (in a geometric sequence, terms get large quickly!)
Sum of the 1\textsuperscript{st} n terms of a geometric sequence

\[ S_n = \frac{a_1 (1 - r^n)}{1 - r} \]
What if $0 < r < 1$ or $-1 < r < 0$?

- Examine an example:
- If $1^{st}$ term = 6 and $r = -1/3$

$$6, -2, \frac{2}{3}, -\frac{2}{9}, \frac{2}{27}, -\frac{2}{81}, \frac{2}{243}, \ldots$$

- Even though the terms are alternating between pos. & neg., their magnitude is getting smaller & smaller
- Imagine infinitely many of these terms: the terms become infinitely small
Find the Sum of an Infinite Geometric Series

• If \(-1<r<1\) and \(r\) not equal zero, then we CAN find the sum, even with infinitely many terms (remember, after a while the terms become infinitely small, thus we can find the sum!)

\[
S^n = \left(\frac{a_1(1-r^n)}{1-r}\right)
\]

• If \(S^n = \left(\frac{a_1(1-r^n)}{1-r}\right)\) and \(n\) is getting very large, then \(r\) raised to the \(n\), recall, is getting very, very small…so small it approaches zero, which allows us to replace \(r\) raised to the \(n\)th power with a zero!

• This leads to: \(S^n = \frac{a_1}{1-r}\)
Repeating decimals can be considered as infinite sums

- Example: Write \(0.34444444\ldots\) as an infinite sum
- Separate the \(0.3\) from the rest of the number:
  - \(0.3444\ldots = 0.3 + 0.044444\ldots\)
  - \(0.044444\ldots = 0.04 + 0.004 + 0.0004 + 0.0004 + \ldots\)
- This is an infinite sum with 1st term = \(0.04\), \(r = 0.1\)
  \[
  S_\infty = \frac{0.04}{1 - 0.1} = \frac{0.04}{0.9} = \frac{4}{90} = \frac{2}{45}
  \]
- \(0.3444\ldots = \frac{3}{10} + \frac{2}{45} = \frac{27}{90} + \frac{4}{90} = \frac{31}{90}\)