5.3 Exponential Functions and Models

- Distinguish between linear and exponential growth
- Model data with exponential functions
- Calculate compound interest
- Use the natural exponential functions in applications
Population Growth by a Constant Number vs by a Constant Percentage

Suppose a population is 10,000 in January 2004. Suppose the population increases by...

- **500 people per year**
  - What is the population in Jan 2005?
    - $10,000 + 500 = 10,500$
  - What is the population in Jan 2006?
    - $10,500 + 500 = 11,000$

- **5% per year**
  - What is the population in Jan 2005?
    - $10,000 + 0.05(10,000) = 10,000 + 500 = 10,500$
  - What is the population in Jan 2006?
    - $10,500 + 0.05(10,500) = 10,500 + 525 = 11,025$
Suppose a population is 10,000 in Jan 2004. Suppose the population increases by 500 per year. What is the population in ....

- Jan 2005?
  - 10,000 + 500 = 10,500
- Jan 2006?
  - 10,000 + 2(500) = 11,000
- Jan 2007?
  - 10,000 + 3(500) = 11,500
- Jan 2008?
  - 10,000 + 4(500) = 12,000
Suppose a population is 10,000 in Jan 2004 and increases by 500 per year.

- Let \( t \) be the number of years after 2004. Let \( P(t) \) be the population in year \( t \). What is the symbolic representation for \( P(t) \)? We know...
  - Population in 2004 = \( P(0) = 10,000 + 0(500) \)
  - Population in 2005 = \( P(1) = 10,000 + 1(500) \)
  - Population in 2006 = \( P(2) = 10,000 + 2(500) \)
  - Population in 2007 = \( P(3) = 10,000 + 3(500) \)
  - Population \( t \) years after 2004 = \( P(t) = 10,000 + t(500) \)
Population is 10,000 in 2004; increases by 500 per yr \( P(t) = 10,000 + t(500) \)

- \( P \) is a **linear** function of \( t \).
- What is the slope?
  - 500 people/year
- What is the \( y \)-intercept?
  - number of people at time 0 (the year 2004) = 10,000

When \( P \) increases by a constant number of people per year, \( P \) is a linear function of \( t \).
Suppose a population is 10,000 in Jan 2004. More realistically, suppose the population increases by \textbf{5\% per year}.

What is the population in ....

- Jan 2005?
  - \(10,000 + 0.05(10,000) = 10,000 + 500 = 10,500\)

- Jan 2006?
  - \(10,500 + 0.05(10,500) = 10,500 + 525 = 11,025\)

- Jan 2007?
  - \(11,025 + 0.05(11,025) = 11,025 + 551.25 = 11,576.25\)
Suppose a population is 10,000 in Jan 2004 and increases by 5% per year.

- Let $t$ be the number of years after 2004. Let $P(t)$ be the population in year $t$. What is the symbolic representation for $P(t)$? We know…
- Population in 2004 = $P(0) = 10,000$
- Population in 2005 = $P(1) = 10,000 + .05 (10,000) = 1.05(10,000) = 1.05^1(10,000) = 10,500$
- Population in 2006 = $P(2) = 10,500 + .05 (10,500) = 1.05 (10,500) = 1.05 (1.05)(10,000) = 1.05^2(10,000) = 11,025$
- Population $t$ years after 2004 = $P(t) = 10,000(1.05)^t$
Population is 10,000 in 2004; increases by 5% per yr \( P(t) = 10,000 \cdot (1.05)^t \)

- P is an **EXponential** function of t. More specifically, an exponential growth function.
- What is the base of the exponential function?  
  - 1.05
- What is the y-intercept?  
  - number of people at time 0 (the year 2004) = 10,000

When P increases by a constant percentage per year, P is an exponential function of t.
Linear vs. Exponential Growth

- A Linear Function adds a fixed amount to the previous value of $y$ for each unit increase in $x$.
- For example, in $f(x) = 10,000 + 500x$, 500 is added to $y$ for each increase of 1 in $x$.

- An Exponential Function multiplies a fixed amount to the previous value of $y$ for each unit increase in $x$.
- For example, in $f(x) = 10,000 (1.05)^x$, $y$ is multiplied by 1.05 for each increase of 1 in $x$. 
Definition of Exponential Function

• A function represented by

\[ f(x) = Ca^x, \quad a > 0, \text{a not 1, and } C > 0 \]

is an exponential function with base \( a \) and coefficient \( C \).

• If \( a > 1 \), then \( f \) is an exponential growth function
• If \( 0 < a < 1 \), then \( f \) is an exponential decay function
Caution

- Don’t confuse $f(x) = 2^x$ with $f(x) = x^2$
- $f(x) = 2^x$ is an exponential function.
- $f(x) = x^2$ is a polynomial function, specifically a quadratic function.
- The functions and consequently their graphs are very different.
Comparison of Exponential and Linear Functions

\[ y = 10000 \cdot (1.05)^x \]

\[ y = 10000 + 500x \]
## Linear Function

### Equation

\[ y = 10000 + 500x \]

### Table

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
<th>( \frac{\Delta y}{\Delta x} )</th>
<th>( \frac{\Delta y}{\Delta x} )</th>
</tr>
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<tbody>
<tr>
<td>0</td>
<td>10000</td>
<td>0</td>
<td>0</td>
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<tr>
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<td>500</td>
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<td>2</td>
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<td>6</td>
<td>13000</td>
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</table>

**Linear Function - Slope is constant.**
Exponential Function

\[ Y = 10000 \times (1.05)^x \]

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
<th>Ratios of consecutive ( y )-values (corresponding to unit increases in ( x ))</th>
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</thead>
<tbody>
<tr>
<td>0</td>
<td>10,000</td>
<td></td>
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<tr>
<td>1</td>
<td>10,500</td>
<td>( \frac{10500}{10000} = 1.05 )</td>
</tr>
<tr>
<td>2</td>
<td>11,025</td>
<td>( \frac{11025}{10500} = 1.05 )</td>
</tr>
<tr>
<td>3</td>
<td>11,576</td>
<td>( \frac{11576}{11025} = 1.05 )</td>
</tr>
<tr>
<td>4</td>
<td>12,155</td>
<td>( \frac{12155}{11576} = 1.05 )</td>
</tr>
<tr>
<td>5</td>
<td>12,763</td>
<td>( \frac{12763}{12155} = 1.05 )</td>
</tr>
<tr>
<td>6</td>
<td>13,401</td>
<td>( \frac{13401}{12763} = 1.05 )</td>
</tr>
</tbody>
</table>

Note that this constant is the base of the exponential function.

Exponential Function - Ratios of consecutive \( y \)-values (corresponding to unit increases in \( x \)) are constant, in this case 1.05.
Which function is linear and which is exponential?

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
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<tbody>
<tr>
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<td>3/8</td>
</tr>
<tr>
<td>-2</td>
<td>3/4</td>
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<tr>
<td>-1</td>
<td>3/2</td>
</tr>
<tr>
<td>0</td>
<td>3</td>
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<tr>
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<table>
<thead>
<tr>
<th>x</th>
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<td>2</td>
<td>-1</td>
</tr>
<tr>
<td>3</td>
<td>-3</td>
</tr>
</tbody>
</table>

For the linear function, tell the slope and y-intercept. For the exponential function, tell the base and the y-intercept. Write the equation of each.
Which function is linear and which is exponential? continued

<table>
<thead>
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<th>$x$</th>
<th>$y$</th>
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<tbody>
<tr>
<td>-3</td>
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<td>12</td>
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<tr>
<td>3</td>
<td>24</td>
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</table>

$y$ is an exponential function of $x$ because the ratio of consecutive values of $y$ is constant, namely 2. Thus the base is 2. The $y$-intercept is 3. Thus the equation is $y = 3 \cdot 2^x$.
Which function is linear and which is exponential? continued

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
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<tbody>
<tr>
<td>-3</td>
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<tr>
<td>2</td>
<td>-1</td>
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<tr>
<td>3</td>
<td>-3</td>
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</table>

$y$ is a linear function of $x$ because the slope is constant, namely $-2/1 = -2$. The $y$-intercept is 3. Thus the equation is $y = -2x + 3$.
Exponential Growth vs Decay

- Example of exponential growth function
  \[ f(x) = 3 \cdot 2^x \]

- Example of exponential decay function
  \[ f(x) = 3 \cdot \left( \frac{1}{2} \right)^x \]
  \[ f(x) = 3 \cdot 2^{-x} \]
Recall

- In the exponential function

\[ f(x) = Ca^x \]

- If \( a > 1 \), then \( f \) is an exponential growth function
- If \( 0 < a < 1 \), then \( f \) is an exponential decay function
Exponential Growth Function
\[ f(x) = Ca^x \text{ where } a > 1 \]

- Example
- \[ f(x) = 3 \cdot 2^x \]

- Properties of an exponential growth function
  - Domain: \((-\infty, \infty)\)
  - Range: \((0, \infty)\)
  - \(f\) increases on \((-\infty, \infty)\)
  - The negative x-axis is a horizontal asymptote.
  - y-intercept is (0,3).
Exponential Decay Function

\[ f(x) = Ca^x \text{ where } 0 < a < 1 \]

- **Example**
  
  \[ f(x) = 3 \cdot \left( \frac{1}{2} \right)^x \]
  
  \[ f(x) = 3 \cdot 2^{-x} \]

- **Properties of an exponential decay function**
  
  - Domain: \((-\infty, \infty)\)
  - Range: \((0, \infty)\)
  - \(f\) decreases on \((-\infty, \infty)\)
  - The positive \(x\)-axis is a horizontal asymptote.
  - \(y\)-intercept is \((0,3)\).
Example of exponential decay - Carbon-14 dating

- The time it takes for half of the atoms to decay into a different element is called the **half-life** of an element undergoing radioactive decay.
- The half-life of carbon-14 is 5700 years.
- Suppose \( C \) grams of carbon-14 are present at \( t = 0 \). Then after 5700 years there will be \( C/2 \) grams present.
Recall the half-life of carbon-14 is 5700 years.

• Let \( t \) be the number of years.
• Let \( A = f(t) \) be the amount of carbon-14 present at time \( t \).
• Let \( C \) be the amount of carbon-14 present at \( t = 0 \).
• Then \( f(0) = C \) and \( f(5700) = C/2 \).
• Thus two points of \( f \) are \((0, C)\) and \((5700, C/2)\).
• Using the point \((5700, C/2)\) and substituting 5700 for \( t \) and \( C/2 \) for \( A \) in \( A = f(t) = Ca^t \) yields:
  \[
  C/2 = C \cdot a^{5700}
  \]
• Dividing both sides by \( C \) yields: \( 1/2 = a^{5700} \)
Recall the half-life of carbon-14 is 5700 years.

\[ \frac{1}{2} = a^{5700} \]

Raising both sides to the \(1/5700\) power gives

\[ \left( \frac{1}{2} \right)^{\frac{1}{5700}} = a \]

So \( A = f(t) = Ca^t \) becomes

\[ A = f(t) = C \left( \frac{1}{2} \right)^{\frac{t}{5700}} \]

Half-life
Generalizing this

- If a radioactive sample containing $C$ units has a half-life of $k$ years, then the amount $A$ remaining after $x$ years is given by

$$A(x) = C \left( \frac{1}{2} \right)^{\frac{x}{k}}$$
Example of Radioactive Decay

- Radioactive strontium-90 has a half-life of about 28 years and sometimes contaminates the soil. After 50 years, what percentage of a sample of radioactive strontium would remain?

\[
A(x) = C \left( \frac{1}{2} \right)^{\frac{x}{k}}
\]

\[
A(50) = C \left( \frac{1}{2} \right)^{\frac{50}{28}} \approx C(0.2900323465)
\]

Since C is present initially and after 50 years, .29C remains, then 29% remains.
Example of Exponential Growth - Compound Interest

- Suppose $10,000 is deposited into an account which pays 5% interest compounded annually. Then the amount $A$ in the account after $t$ years is:
  \[ A(t) = 10,000 \times (1.05)^t \]

- Note the similarity with: Suppose a population is 10,000 in 2004 and increases by 5% per year. Then the population $P$, $t$ years after 2004 is:
  \[ P(t) = 10,000 \times (1.05)^t \]
Frequencies of Compounding (Adding Interest)

- annually (1 time per year)
- semiannually (2 times per year)
- quarterly (4 times per year)
- monthly (12 times per year)
- daily (365 times per year)
Compound Interest Formula

- If $P$ dollars is deposited in an account paying an annual rate of interest $r$, compounded (paid) $n$ times per year, then after $t$ years the account will contain $A$ dollars, where

$$A = P \left(1 + \frac{r}{n}\right)^{nt}$$
Suppose $1000 is deposited into an account yielding 5% interest compounded at the following frequencies. How much money after \( t \) years?

\[
A = P \left( 1 + \frac{r}{n} \right)^{nt}
\]

- **Annually**
  
  \[A = 1000 \left( 1 + \frac{.05}{1} \right)^{1t} = 1000(1.05)^t\]

- **Semiannually**
  
  \[A = 1000 \left( 1 + \frac{.05}{2} \right)^{2t} = 1000(1.025)^{2t}\]

- **Quarterly**
  
  \[A = 1000 \left( 1 + \frac{.05}{4} \right)^{4t} = 1000(1.0125)^{4t}\]

- **Monthly**
  
  \[A = 1000 \left( 1 + \frac{.05}{12} \right)^{12t} = 1000(1.00416)^{12t}\]
The Natural Exponential Function

- The function \( f \), represented by \( f(x) = e^x \)

is the natural exponential function where

\[ e \approx 2.718281828 \]
Continuously Compounded Interest

- If a principal of $P$ dollars is deposited in an account paying an annual rate of interest $r$ (expressed in decimal form), compounded continuously, then after $t$ years the account will contain $A$ dollars, where

$$A = Pe^{rt}$$
Example

• Suppose $100 is invested in an account with an interest rate of 8% compounded continuously. How much money will there be in the account after 15 years?

\[ A = Pe^{rt} \]

\[ A = $100 \; e^{0.08 \times 15} \]

\[ A = $332.01 \]
5.4 Logarithmic Functions and Models

- Evaluate the common logarithm function
- Solve basic exponential and logarithmic equations
- Evaluate logarithms with other bases
- Solve general exponential and logarithmic equations
Common Logarithm

- The common logarithm of a positive number $x$, denoted $\log x$, is defined by
  \[ \log x = k \text{ if and only if } x = 10^k \]
  where $k$ is a real number.

- The function given by $f(x) = \log x$ is called the common logarithm function.
Evaluate each of the following.

- \( \log_{10} 1 \) because \( 10^1 = 10 \)
- \( \log_{10} 10 \) because \( 10^2 = 100 \)
- \( \log_{10} 100 \) because \( 10^3 = 1000 \)
- \( \log_{10} 1000 \) because \( 10^4 = 10000 \)
- \( \log_{10} 10000 \) because \( 10^{-1} = 1/10 \)
- \( \log_{10} (1/10) \) because \( 10^{-2} = 1/100 \)
- \( \log_{10} (1/100) \) because \( 10^{-3} = 1/1000 \)
- \( \log_{10} (1/1000) \) because \( 10^0 = 1 \)
Graph of $f(x) = \log x$

Note that the graph of $y = \log x$ is the graph of $y = 10^x$ reflected through the line $y = x$. This suggests that these are inverse functions.
The Inverse of $y = \log x$

- Note that the graph of $f(x) = \log x$ passes the horizontal line test so it is a 1-1 function and has an inverse function.
- Find the inverse of $y = \log x$
- Using the definition of common logarithm to solve for $x$ gives
  - $x = 10^y$
- Interchanging $x$ and $y$ gives
  - $y = 10^x$
- So yes, the inverse of $y = \log x$ is $y = 10^x$
Inverse Properties of the Common Logarithm

- Recall that \( f^{-1}(x) = 10^x \) given \( f(x) = \log x \)

- Since \((f \circ f^{-1})(x) = x\) for every \(x\) in the domain of \(f^{-1}\)
  - \(\log(10^x) = x\) for all real numbers \(x\).

- Since \((f^{-1} \circ f)(x) = x\) for every \(x\) in the domain of \(f\)
  - \(10^{\log x} = x\) for any positive number \(x\)
Solving Exponential Equations Using The Inverse Property $\log(10^x) = x$

- Solve the equation $10^x = 35$
- Take the common log of both sides
  - $\log 10^x = \log 35$
- Using the inverse property $\log(10^x) = x$ this simplifies to
  - $x = \log 35$
- Using the calculator to estimate $\log 35$ we have
  - $x \approx 1.54$
Solving Logarithmic Equations Using The Inverse Property $10^{\log x} = x$

- Solve the equation $\log x = 4.2$
- Exponentiate each side using base 10
  - $10^{\log x} = 10^{4.2}$
- Using the inverse property $10^{\log x} = x$ this simplifies to
  - $x = 10^{4.2}$
- Using the calculator to estimate $10^{4.2}$ we have
  - $x \approx 15848.93$
Definition of Logarithm With Base $a$

- The logarithm with base $a$ of a positive number $x$, denoted by $\log_a x$ is defined by $\log_a x = k$ if and only if $x = a^k$
  where $a > 0$, $a \neq 1$, and $k$ is a real number.

- The function given by $f(x) = \log_a x$ is called the logarithmic function with base $a$. 
Practice with the Definition

Practice Questions:

• \( \log_b c = d \) means \( \quad \)
• \( p = \log_y m \) means \( \quad \)
• True or false:
  • True or false: \( \log_2 8 = 3 \)
  • True or false: \( \log_5 25 = 2 \)
  • True or false: \( \log_{25} 5 = 1/2 \)
  • True or false: \( \log_4 8 = 2 \)

What is the value of \( \log_4 8 \)?

It is 3/2 because \( 4^{3/2} = 8 \)

Answers:

• \( b^d = c \)
• \( y^p = m \)
• True because \( 2^3 = 8 \)
• True because \( 5^2 = 25 \)
• True because \( 25^{1/2} = 5 \)
• False because \( 4^2 = 16 \) not 8
Practice Evaluating Logarithms

Evaluate

• \( \log_6 36 \)
• \( \log_{36} 6 \)
• \( \log_2 32 \)
• \( \log_{32} 2 \)
• \( \log_6 (1/36) \)
• \( \log_2 (1/32) \)
• \( \log 100 \)
• \( \log (1/10) \)
• \( \log 1 \)

Answers:

• 2 because \( 6^2 = 36 \)
• 1/2 because \( 36^{(1/2)} = 6 \)
• 5 because \( 2^5 = 32 \)
• 1/5 because \( 32^{(1/5)} = 2 \)
• –2 because \( 6^{-2} = 1/36 \)
• –5 because \( 2^{-5} = 1/32 \)
• 2 because \( 10^2 = 100 \)
• –1 because \( 10^{-1} = 1/10 \)
• 0 because \( 10^0 = 1 \)
Calculators and logarithms

- The TI-83 evaluates base 10 logarithms and base e logarithms.
- Base 10 logs are called common logs.
  - \( \log x \) means \( \log_{10}x \).
  - Notice the log button on the calculator.
- Base e logs are called natural logs.
  - \( \ln x \) means \( \log_e x \).
  - Notice the ln button on the calculator.
Evaluate each of the following without calculator. Then check with calculator.

- \( \ln(e) \)
  - \( \ln(e) = \log_e e = 1 \) since \( e^1 = e \)

- \( \ln(e^2) \)
  - \( \ln(e^2) = \log_e (e^2) = 2 \) since 2 is the exponent that goes on \( e \) to produce \( e^2 \).

- \( \ln(1) \)
  - \( \ln(1) = \log_e 1 = 0 \) since \( e^0 = 1 \)

- \( \ln(\sqrt{e}) \)
  - \( 1/2 \) since \( 1/2 \) is the exponent that goes on \( e \) to produce \( e^{1/2} \).
The Inverse of \( y = \log_a x \)

- Note that the graph of \( f(x) = \log_a x \) passes the horizontal line test so it is a 1-1 function and has an inverse function.
- Find the inverse of \( y = \log_a x \)
- Using the definition of common logarithm to solve for \( x \) gives
  - \( x = a^y \)
- Interchanging \( x \) and \( y \) gives
  - \( y = a^x \)
- So the inverse of \( y = \log_a x \) is \( y = a^x \)
Inverse Properties of Logarithms With Base \( a \)

- Recall that \( f^{-1}(x) = a^x \) given \( f(x) = \log_a x \)

- Since \((f \circ f^{-1})(x) = x\) for every \( x \) in the domain of \( f^{-1} \)
  - \( \log_a(a^x) = x \) for all real numbers \( x \).

- Since \((f^{-1} \circ f)(x) = x\) for every \( x \) in the domain of \( f \)
  - \( a^{\log_a x} = x \) for any positive number \( x \).
Solving Exponential Equations Using The Inverse Property $\log_a(a^x) = x$

- Solve the equation $4^x = 1/64$
- Take the log of both sides to the base 4
  - $\log_4(4^x) = \log_4(1/64)$
- Using the inverse property $\log_a(a^x) = x$ this simplifies to
  - $x = \log_4(1/64)$
- Since $1/64$ can be rewritten as $4^{-3}$
  - $x = \log_4(4^{-3})$
- Using the inverse property $\log_a(a^x) = x$ this simplifies to
  - $x = -3$
Solving Exponential Equations Using The Inverse Property \( \log_a (a^x) = x \)

- Solve the equation \( e^x = 15 \)
- Take the log of both sides to the base \( e \)
  - \( \ln(e^x) = \ln(15) \)
- Using the inverse property \( \log_a (a^x) = x \) this simplifies to
  - \( x = \ln 15 \)
- Using the calculator to estimate \( \ln 15 \)
  - \( x \approx 2.71 \)
Solving Logarithmic Equations Using
The Inverse Property $a^{\log_a x} = x$

- Solve the equation $\ln x = 1.5$
- Exponentiate both sides using base $e$
  - $e^{\ln x} = e^{1.5}$
- Using the inverse property $a^{\log_a x} = x$ this simplifies to
  - $x = e^{1.5}$
- Using the calculator to estimate $e^{1.5}$
  - $x \approx 4.48$
Recall from section 5.3

Graph of $f(x) = a^x$ where $a > 1$

Graph of $f(x) = a^x$ where $0 < a < 1$

Using the fact that the graph of a function and its inverse are symmetric with respect to the line $y = x$, the graph $f^{-1}(x) = \log_a x$ for the two types of exponential functions listed above. Looking at the two resulting graphs, what is the domain of a logarithmic function? What is the range of a logarithmic function?
Graph of $f(x) = a^x$ where $a > 1$

Graph of $f(x) = a^x$ where $0 < a < 1$

Superimpose graphs of the inverses of the functions above similar to Figure 5.58 on page 422