COLLEGE ALGEBRA

with Modeling and Visualization

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THIRD EDITION
Sequences

- Understand basic concepts about sequences
- Learn how to represent sequences
- Identify and use arithmetic sequences
- Identify and use geometric sequences
Introduction

- A sequence is a function that computes an ordered list.

Example

If an employee earns $12 per hour, the function \( f(n) = 12n \) generates the terms of the sequence \( 12, 24, 36, 48, 60, \ldots \) when \( n = 1, 2, 3, 4, 5, \ldots \)

**SEQUENCE**

An **infinite sequence** is a function that has the set of natural numbers as its domain. A **finite sequence** is a function with domain \( D = \{1, 2, 3, \ldots, n\} \), for some fixed natural number \( n \).
Sequences

- Instead of letting $y$ represent the output, it is common to write $a_n = f(n)$, where $n$ is a natural number in the domain of the sequence. The terms of a sequence are

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

- The first term is $a_1 = f(1)$, the second term is $a_2 = f(2)$ and so on. The $n$th term or general term of a sequence is $a_n = f(n)$. 
Example

Write the first four terms $a_1$, $a_2$, $a_3$, $a_4$,… of each sequence, where $a_n = f(n)$,

a) $f(n) = 5n + 3$  

Solution

a) $a_1 = f(1) = 5(1) + 3 = 8$
   $a_2 = f(2) = 5(2) + 3 = 13$
   $a_3 = f(3) = 5(3) + 3 = 18$
   $a_4 = f(4) = 5(4) + 3 = 23$

b) $f(n) = (4)^{n-1} + 2$

b) $a_1 = f(1) = (4)^{1-1} + 2 = 2$
   $a_2 = f(2) = (4)^{2-1} + 2 = 6$
   $a_3 = f(3) = (4)^{3-1} + 2 = 18$
   $a_4 = f(4) = (4)^{4-1} + 2 = 66$
Recursive Sequence

- With a recursive sequence, one or more previous terms are used to generate the next term.
- The terms $a_1$ through $a_{n-1}$ must be found before $a_n$ can be found.

Example

a) Find the first four terms of the recursive sequence that is defined by $a_n = 3a_{n-1} + 5$ and $a_1 = 4$, where $n \geq 2$.

b) Graph the first 4 terms of the sequence.
Example continued

Solution

a) *Numerical Representation*

\[ a_1 = 4 \]
\[ a_2 = 3a_1 + 5 = 3(4) + 5 = 17 \]
\[ a_3 = 3a_2 + 5 = 3(17) + 5 = 56 \]
\[ a_4 = 3a_3 + 5 = 3(56) + 5 = 173 \]

The first four terms are 4, 17, 56, and 173.

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_n )</td>
<td>4</td>
<td>17</td>
<td>56</td>
<td>173</td>
</tr>
</tbody>
</table>
Solution continued

Graphical Representation

b) To represent these terms graphically, plot the points. Since the domain of the graph only contains natural numbers, the graph of the sequence is a scatterplot.
Arithmetic Sequences

**Example**

An employee receives 10 vacation days per year. Thereafter the employee receives an additional 2 days per year with the company. The amount of vacation days after \( n \) years with the company is represented by

\[
 f(n) = 2n + 10, \text{ where } f \text{ is a linear function.}
\]

How many vacation days does the employee have after 14 years? (Assume no rollover of days.)

**Solution**

\[
 f(14) = 2(14) + 10 = 38 \text{ days of vacation.}
\]
Arithmetic Sequence

- An arithmetic sequence can be defined recursively by $a_n = a_{n-1} + d$, where $d$ is a constant. Since $d = a_n - a_{n-1}$ for each valid $n$, $d$ is called the common difference. If $d = 0$, then the sequence is a constant sequence.

- A finite arithmetic sequence is similar to an infinite arithmetic sequence except its domain is $D = \{1, 2, 3, \ldots, n\}$, where $n$ is a fixed natural number.

- Since an arithmetic sequence is a linear function, it can always be represented by $f(n) = dn + c$, where $d$ is the common difference and $c$ is a constant.
Example

Find a general term $a_n = f(n)$ for the arithmetic sequence; $a_1 = 4$ and $d = -3$.

Solution

Let $f(n) = dn + c$.
Since $d = -3$, $f(n) = -3n + c$.

$$a_1 = f(1) = -3(1) + c = 4 \quad \text{or} \quad c = 7$$

Thus $a_n = -3n + 7$. 
Terms of an Arithmetic Sequence

**nth TERM OF AN ARITHMETIC SEQUENCE**

In an arithmetic sequence with first term $a_1$ and common difference $d$, the $n$th term, $a_n$, is given by

$$a_n = a_1 + (n - 1)d.$$ 

**Example**

Find a symbolic representation (formula) for the arithmetic sequence given by $6, 10, 14, 18, 22,...$

**Solution**

The first term is 6. Successive terms can be found by adding 4 to the previous term. $a_1 = 6$ and $d = 4$

$$a_n = a_1 + (n - 1)d$$

$$= 6 + (n - 1)(4)$$

$$= 4n + 2$$
Geometric Sequences

- Geometric sequences are capable of either rapid growth or decay.

Examples
- Population
- Salary
- Automobile depreciation

**INFINITE GEOMETRIC SEQUENCE**

An infinite geometric sequence is a function defined by $f(n) = cr^{n-1}$, where $c$ and $r$ are nonzero constants. The domain of $f$ is the set of natural numbers.
Example

Find a general term \( a_n \) for the geometric sequence; \( a_3 = 18 \) and \( a_6 = 486 \).

Solution

Find \( a_n = cr^{n-1} \) so that \( a_3 = 18 \) and \( a_6 = 486 \).

Since \( \frac{a_6}{a_3} = \frac{cr^{6-1}}{cr^{3-1}} = \frac{r^5}{r^2} = r^3 \) and \( \frac{a_6}{a_3} = \frac{486}{18} = 27 \),

\[ r^3 = 27 \text{ or } r = 3. \]

So \( a_n = c(3)^{n-1} \).
Therefore \( a_3 = c(3)^{3-1} = 18 \) or \( c = 2 \).
Thus \( a_n = 2(3)^{n-1} \).
Understand basic concepts about series
Identify and find the sum of arithmetic series
Identify and find the sum of geometric series
Learn and use summation notation
Introduction

- A series is the summation of the terms in a sequence.
- Series are used to approximate functions that are too complicated to have a simple formula.
- Series are instrumental in calculating approximations of numbers like $\pi$ and $e$.

**SERIES**

A finite series is an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n,$$

and an infinite series is an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots.$$
Infinite Series

- An infinite series contains many terms.
- Sequence of partial sums:
  \[
  S_1 = a_1 \\
  S_2 = a_1 + a_2 \\
  S_3 = a_1 + a_2 + a_3 \\
  \vdots \\
  S_n = a_1 + a_2 + a_3 + \cdots + a_n
  \]

- If \( S_n \) approaches a real number \( S \) as \( n \to \infty \), then the sum of the infinite series is \( S \).
- Some infinite series do not have a sum \( S \).
Example

For $a_n = 3n - 1$, calculate $S_5$.

Solution

Since $S_5 = a_1 + a_2 + a_3 + a_4 + a_5$, start calculating the first five terms of the sequence $a_n = 3n - 1$.

$\begin{align*}
a_1 &= 3(1) - 1 = 2 \\
a_2 &= 3(2) - 1 = 5 \\
a_3 &= 3(3) - 1 = 8 \\
a_4 &= 3(4) - 1 = 11 \\
a_5 &= 3(5) - 1 = 14 \\
\end{align*}$

Thus $S_5 = 2 + 5 + 8 + 11 + 14 = 40$
Arithmetic Series

- Summing the terms of a arithmetic sequence results in an arithmetic series.

**SUM OF THE FIRST \( n \) TERMS OF AN ARITHMETIC SEQUENCE**

The sum of the first \( n \) terms of an arithmetic sequence, denoted \( S_n \), is found by averaging the first and \( n \)th terms and then multiplying by \( n \). That is,

\[
S_n = a_1 + a_2 + a_3 + \cdots + a_n = n \left( \frac{a_1 + a_n}{2} \right).
\]
Arithmetic Series continued

Since \( a_n = a_1 + (n - 1)d \), \( S_n \) can also be written in the following way.

\[
S_n = n \left( \frac{a_1 + a_n}{2} \right) 
\]

\[
= \frac{n}{2} \left( a_1 + a_1 + (n-1)d \right) 
\]

\[
= \frac{n}{2} \left( 2a_1 + (n-1)d \right) 
\]
Example

Use the formula to find the sum of the arithmetic series $4 + 7 + 10 + \cdots + 58$.

Solution

The series has $n = 19$ terms with $a_1 = 4$ and $a_{19} = 58$. We can then use the formula to find the sum.

$$S_n = n \left( \frac{a_1 + a_n}{2} \right)$$

$$S_{19} = 19 \left( \frac{4 + 58}{2} \right) = 589$$
Example

A worker has a starting annual salary of $45,000 and receives a $2500 raise each year. Calculate the total amount earned over 5 years.

Solution

The arithmetic sequence describing the salary during year $n$ is computed by

$$a_n = 45,000 + 2500(n - 1).$$

The first and fifth year’s salaries are

$$a_1 = 45,000 + 2500(1 - 1) = 45,000$$

$$a_5 = 45,000 + 2500(5 - 1) = 55,000$$
Solution continued

Thus the total amount earned during this 5-year period is

\[ S_5 = 5 \left( \frac{45,000 + 55,000}{2} \right) = $250,000. \]

The sum can also be found using

\[ S_n = \frac{n}{2} \left( 2a_1 + (n - 1)d \right). \]

\[ S_5 = \frac{5}{2} \left( 2 \cdot 45,000 + (5 - 1) \cdot 2500 \right) = $250,000. \]
The sum of the terms of a geometric sequence is called a geometric series.

**SUM OF THE FIRST \( n \) TERMS OF A GEOMETRIC SEQUENCE**

If a geometric sequence has first term \( a_1 \) and common ratio \( r \), then the sum of the first \( n \) terms is given by

\[
S_n = a_1 \left( \frac{1 - r^n}{1 - r} \right),
\]

provided \( r \neq 1 \).
Annuities

- An *annuity* is a sequence of deposits made at equal periods of time.
- After *n* years the amount is given by
  \[ A_0 + A_0 (1+i) + A_0 (1+i)^2 + \cdots + A_0 (1+i)^{n-1}. \]

- This is a geometric series with first term \( a_1 = A_0 \) and common ratio \( r = (1 + i) \). The sum of the first *n* terms is given by
  \[ S_n = A_0 \left( \frac{1-(1+i)^n}{1-(1+i)} \right) = A_0 \left( \frac{(1+i)^n - 1}{i} \right). \]
Example

A 30-year-old employee deposits $4000 into an account at the end of each year until age 65. If the interest rate is 8%, find the future value of the annuity.

Solution Let $A_0 = 4000$, $i = 0.08$, and $n = 35$. The future value of the annuity is given by

$$S_n = A_0 \left( \frac{(1+i)^n - 1}{i} \right)$$

$$= 4000 \left( \frac{(1+0.08)^{35} - 1}{0.08} \right)$$

$$= $689,267.$
Infinite Geometric Series

**SUM OF AN INFINITE GEOMETRIC SEQUENCE**

The sum of the infinite geometric sequence with first term \( a_1 \) and common ratio \( r \) is given by

\[
S = \frac{a_1}{1 - r},
\]

provided \( |r| < 1 \). If \( |r| \geq 1 \), then this sum does not exist.
Summation Notation

- *Summation notation* is used to write series efficiently. The symbol $\sum$, sigma, indicates the sum.

- The letter $k$ is called the *index of summation*. The numbers 1 and $n$ represent the subscripts of the first and last term in the series. They are called the *lower limit* and *upper limit* of the summation, respectively.
Example

Evaluate each series.

a) \[ \sum_{k=1}^{3} 4k = 4 + 8 + 12 = 24 \]

b) \[ \sum_{k=1}^{3} 4 = 4 + 4 + 4 = 12 \]

c) \[ \sum_{k=1}^{6} (3k + 6) = (3(1) + 6) + (3(2) + 6) + (3(3) + 6) + (3(4) + 6) + (3(5) + 6) + (3(6) + 6) \]
\[ = 9 + 12 + 15 + 18 + 21 + 24 = 99 \]
Properties for Summation Notation

PROPERTIES FOR SUMMATION NOTATION

Let \( a_1, a_2, a_3, \ldots, a_n \) and \( b_1, b_2, b_3, \ldots, b_n \) be sequences, and \( c \) be a constant.

1. \( \sum_{k=1}^{n} ca_k = c \sum_{k=1}^{n} a_k \)

2. \( \sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k \)

3. \( \sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k \)

4. \( \sum_{k=1}^{n} c = nc \)

5. \( \sum_{k=1}^{n} k = \frac{n(n + 1)}{2} \)

6. \( \sum_{k=1}^{n} k^2 = \frac{n(n + 1)(2n + 1)}{6} \)
Example

Use properties for summation notation to find each sum.

a) $\sum_{k=1}^{18} 4k$

b) $\sum_{k=1}^{11} 4k^2 - 6$

Solution

a) $\sum_{k=1}^{18} 4k = 4 \sum_{k=1}^{18} k$

$= 4 \times \frac{18(18+1)}{2}$

$= 684$

b) $\sum_{k=1}^{11} 4k^2 - 6 = \sum_{k=1}^{11} 4k^2 - \sum_{k=1}^{11} 6$

$= 4 \sum_{k=1}^{11} k^2 - \sum_{k=1}^{11} 6$

$= 4 \times \frac{11(11+1)(2\times11+1)}{6} - 11(6)$

$= 1958$