3.4 Zeros of Polynomials
In this section, we study methods for finding zeros of polynomial functions.

**Remember:** the relationship among zeros, roots, and x-intercepts.
The zeros of a function are the roots, or solutions, of the equation \( f(x) = 0 \).
The real zeros, or real roots, are the x-intercepts of the graph of \( f \).

❖ **Rational Zero Theorem**
This theorem provides us with a tool that we can use to make a list of all possible rational zeros of a polynomial function.

<table>
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<th>The Rational Zero Theorem</th>
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| If \( f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 \) has integer coefficients and \( \frac{p}{q} \) is a rational zero of \( f \), then \( p \) is a factor of the constant term, \( a_0 \), and \( q \) is a factor of the leading coefficient, \( a_n \).

Find all the possible rational zeros:
1.) \( p \): List all the integers that are factors of the constant term.
2.) \( q \): List all the integers that are factors of the leading coefficient.
3.) Possible rational zeros = \( \frac{\text{Factors of the constant term}}{\text{Factors of the leading coefficient}} = \frac{p}{q} \)

Ex. Use the Rational Zero Theorem to list all possible rational zeros for the given function.
\[ f(x) = -4x^4 + 5x^3 - 7x^2 - 6x + 8 \]

\( p \): ____________________________ (factors of 8)

\( q \): ____________________________ (factors of -4)

\( \frac{p}{q} \): ____________________________
Find all real zeros of a polynomial function \( f(x) \):

**Step 1:** Find all possible rational zeros \( \pm \frac{p}{q} \).

**Step 2:** Graph the function and determine which zeros (x-intercepts on the graph) to use in synthetic division.

**Step 3:** Perform synthetic division to determine which possible zeros yield a remainder of zero. If the degree of a polynomial is 3 or higher, continue to use synthetic division (repeat Step 2 to Step 3) until another zero is found.

**Step 4:** Rewrite the function as a product of factors, linear and quadratic. Zeros of the quadratic factor are found by factoring, the quadratic formula, or the square root property.

**Step 5:** Solve \( f(x) = 0 \).

**Properties of Roots of Polynomial Equations**

1. If a polynomial equation is of degree \( n \), then counting multiple roots separately, the equation has \( n \) roots.

2. If \( a + bi \) is a root of a polynomial equation with real coefficients \( (b \neq 0) \), then the imaginary number \( a - bi \) is also a root. Imaginary roots, if they exist, occur in conjugate pairs.

**The Fundamental Theorem of Algebra**

If \( f(x) \) is a polynomial of degree \( n \geq 1 \) with complex coefficients, then \( f(x) \) has at least one complex zero.

Ex. Find all zeros of \( f(x) = 2x^4 + 3x^3 - 15x^2 - 32x - 12 \).
Ex. (#22) Find all zeros of \( f(x) = 7x^3 - x^2 - 21x + 3 \).

Ex. (#38) A polynomial \( f(x) \) and one or more of its zeros is given.

\[
f(x) = 2x^5 - 5x^4 - 4x^3 - 22x^2 + 50x + 75; \quad -1-2i \text{ and } \frac{5}{2} \text{ are zeros}
\]

(a) Find all the zeros.
(b) Factor \( f(x) \) as a product of linear factors.
(c) Solve \( f(x) = 0 \) the equation.